

# Rigidity percolation on the square lattice

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**Abstract.** – The square lattice with central forces between nearest neighbors is isostatic with a subextensive number of floppy modes. It can be made rigid by the random addition of next-nearest neighbor bonds. This constitutes a rigidity percolation transition which we study analytically by mapping it to a connectivity problem of two-colored random graphs. We derive an exact recurrence equation for the probability of having a rigid percolating cluster and solve it in the infinite volume limit. From this solution we obtain the rigidity threshold as a function of system size, and find that, in the thermodynamic limit, there is a mixed first-order-second-order rigidity percolation transition at the isostatic point.

**Introduction.** – Central force rigidity percolation describes how a system of lattice sites can become rigid by randomly populating the bonds with springs or struts that can transmit central forces between sites. It features a mechanical critical point at which an infinite rigid cluster emerges and the system gains rigidity [1–3]. Comparing to the analog scalar problem of connectivity percolation, in which an infinite connected cluster emerges and the system becomes conducting by populating bonds, rigidity percolation is a *vector* problem whose basic degrees of freedom are the  $d$ -dimensional position vectors of the lattice sites. Another important character of rigidity percolation is its *long-range* nature, that adding a bond at one place in the network could affect the rigidity of regions arbitrarily far away from that bond [2, 3], which makes rigidity percolation a challenging problem for theoretical study. Numerical simulations on rigidity percolation revealed highly nontrivial physics. In two dimensions, numerical simulations using the “pebble game” algorithm on generic networks strongly suggest that the rigidity percolation is second order [2–4]. However the possibility of a weakly first order transition [5, 6] is not completely ruled out due to finite size effects in the simulations. In three dimensions, numerical simulations using the “pebble game” algorithm found rigidity percolation to be first order [7].

One special class of rigidity percolation occurs on periodic isostatic lattices with the nearest-neighbor (NN) bonds already present from the start [8]. These lattices are

at the onset of mechanical rigidity (isostaticity) because they have equal numbers of degrees of freedom and constraints in the bulk, which for central forces corresponds to a coordination number  $z = 2d$ , where  $d$  is the dimensionality of the system [9]. Examples are the two-dimensional square and kagome lattices and the three-dimensional cubic lattice. In a finite isostatic lattice, sites on the boundary have coordination number lower than  $2d$ , which gives rise to a subextensive number of deformation modes in which none of the bonds change length. These so-called floppy modes do not cost any elastic energy, and are extended across the lattice: the system cannot be macroscopically rigid unless all floppy modes are somehow removed. To restore rigidity, one can randomly add some of the next-nearest-neighbor (NNN) bonds [4, 5, 10–12]. This constitutes a special kind of rigidity percolation problem because all lattice sites are already connected and one only needs to remove a subextensive number of floppy modes. One therefore expects that a subextensive number of NNN bonds is enough to provide rigidity.

In this Letter, we investigate rigidity percolation on the square lattice, which has coordination number  $z = 4 = 2d$ , and is isostatic [8, 12]. We develop an analytical method to calculate the probability  $\mathcal{F}(L, p)$  that a size  $L \times L$  square lattice is made rigid by populating the NNN bonds with probability  $p$ , and thus we arrive at the rigidity threshold  $p_R(L)$  that half of the realizations are rigid, i.e.,  $\mathcal{F}(L, p_R(L)) = 1/2$ . Surprisingly, we find that the intu-

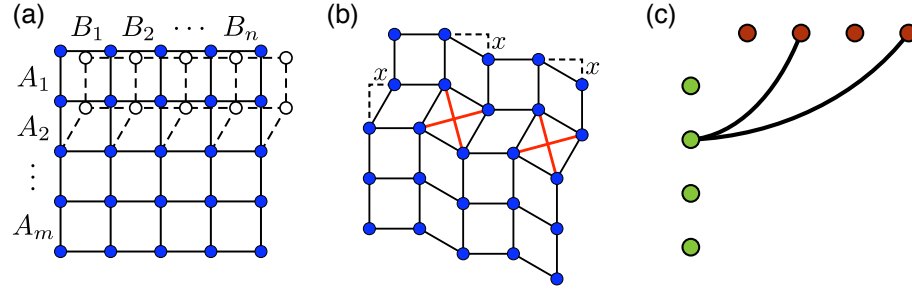


Fig. 1: (a) The variables  $A_i$  and  $B_i$  in which we express the zero energy deformations of the square lattice (before adding NNN bonds) are the relative displacements of entire neighboring rows and columns, respectively. The effect of a nonzero  $A_2$  is illustrated. (b) Adding a NNN bond at location  $(i, j)$  corresponds to constraining the variables  $A_i$  and  $B_j$  to be equal, illustrated using  $B_2 = A_2 = B_4 = x$ . (c) Graph representation of the network shown in (b), where each variable is represented by a node (light green and dark red for the  $A_i$  and  $B_i$ , respectively) and each NNN bond by a link between the corresponding nodes.

itive argument used in earlier works [4, 5, 11], that having at least one NNN bond in each row and column should provide rigidity, presents a necessary but insufficient condition for rigidity of square lattices. In fact, one can add as few as  $L - 1$  NNN bonds to an  $L \times L$  square lattice to make the “one-per-row” condition satisfied, but according to the Maxwell counting [9] there are still  $L - 2$  floppy modes.

To obtain the rigidity threshold, we map the problem into a connectivity problem of two-colored (also known as *bipartite*) random graphs with  $L - 1$  nodes of each color, for which the number of connected clusters corresponds to the number of floppy modes in the square lattice. We derive a recurrence relation for the probability that all nodes of the graph are connected into one cluster, which corresponds to the probability that the entire square lattice is rigid. We solve this equation asymptotically in the large  $L$  limit, and show that, (i) for nonzero  $p$ ,  $\mathcal{F}(L, p) \sim 1 - 2L(1 - p)^L$ , (ii) the rigidity threshold  $p_R$  approaches zero when  $L \rightarrow \infty$  as  $p_R(L) \sim \ln L/L + \mathcal{O}(1/L)$ . We confirm these results using numerical simulations.

Consider a regular square lattice with  $L$  sites per row or column. According to Maxwell [9], the number of floppy modes can be found by subtracting the number of constraints from the number of degrees of freedom (usually, the global translations and rotations are subtracted as well, leaving only the nontrivial floppy modes). Thus the number of floppy modes in the  $L \times L$  square lattice is  $2L^2 - 2L(L - 1) - 3 = 2L - 3$  which can be understood as follows. Each of the rows or columns of square plaquettes can be deformed into rhombi without changing the shape of the plaquettes in other rows or columns, as shown in fig. 1a. These deformations correspond to linearly independent floppy modes of the lattice. The total number of such rows and columns of plaquettes is  $2L - 2$ . Note that any pure shear deformation can be obtained using linear combinations of these modes, which is why the shear modulus is zero<sup>1</sup>. Because the plaquette deformations concern

*relative* displacements of neighboring rows or columns, the space they span does not contain the global translations (but global rotation is included).

To remove these floppy modes and restore shear rigidity, we can add  $2L - 3$  NNN bonds to, for example, the left and the lower boundary plaquettes of the lattice. However, for the case of randomly populating each NNN bond with probability  $p$ , some of the NNN bonds could be redundant, and thus one may need more than  $2L - 3$  NNN bonds to restore the shear rigidity. Determining the rigidity of a disordered network of springs always involves such counting arguments comparing the number of degrees of freedom to the number of *independent* constraints, and because of the disorder it is generally nontrivial to properly keep track of which constraints are truly independent, which makes rigidity percolation difficult theoretically [2, 3]. Below, we will derive simple rules for the removal of these floppy modes that make rigidity percolation on the square lattice tractable.

The spring networks obtained through the addition of NNN bonds, while the NN bonds of the square lattice are already present from the start, are fundamentally different from the classical case of rigidity percolation on the triangular lattice, where no bonds are present when  $p = 0$ . Most strikingly, because all sites are already connected, the probability that a site belongs to the rigid cluster equals one as soon as  $p$  is large enough to have a spanning rigid cluster. Thus, in the thermodynamic limit, the order parameter, defined as the probability that an arbitrarily chosen site belongs to the percolating rigid cluster, jumps from 0 to 1 at the transition, characterizing a first order transition. On the other hand, the system displays a diverging isostatic length scale  $l^* \sim 1/p$  and a vanishing shear modulus  $\mu \sim p^2$  [12], which are characters of a second order transition. The discontinuous change in the order parameter and the continuous change in the shear modulus can be heuristically understood by realizing that

elastic moduli,  $C_{11}$ ,  $C_{12}$  and  $C_{44}$ . With NN bonds only we have  $C_{12} = C_{44} = 0$ , and in this paper we refer to  $C_{44}$  as shear modulus for convenience [13].

<sup>1</sup>Due to anisotropy, strictly speaking, the square lattice has three

the addition of a small number of bonds restores the rigidity of the whole system, thus these bonds carry all stress applied to the system, resulting in a vanishingly small shear modulus [5]. This feature of a mixed first-order-second-order phase transition resembles the jamming transition in packings of frictionless spheres, which features a jump in the average coordination number from zero to the isostatic value, as well as power law scalings of the elastic moduli and the isostatic length scale [14, 15]. The connection between the square lattice and the jamming transition has been discussed in Refs. [8, 12].

**The probability of rigid configurations.** – A square lattice of  $(m + 1) \times (n + 1)$  sites with randomly added next nearest neighbor bonds is rigid if the only zero-energy modes are the global translations and rotations. To calculate the probability that this is the case, we consider the space of floppy modes of the original square lattice, which is a subspace of the space of all modes of the lattice. In particular, in linear elasticity the space of the floppy modes is the null space of the dynamical matrix of the lattice. To exclude the trivial global translational degrees of freedom, we choose to project this space of floppy modes into the basis of deformations of the square plaquettes in each row or column into rhombi, as discussed earlier and shown in fig. 1a, or more precisely, the relative horizontal (vertical) displacements between neighboring rows (columns). The dimension of this space is  $m + n$  with the global rotation included. This leads to a description in terms of relative row-displacements  $A_1, A_2, \dots, A_m$  and relative column-displacements  $B_1, B_2, \dots, B_n$ .

As shown in fig. 1a, exciting the floppy mode labelled by  $A_2$  turns all squares on the corresponding lattice row into rhombi. To make the system rigid, all floppy modes have to be constrained, so that one needs to have at least one NNN bond on each row and on each column. However, as will soon become clear, this necessary condition for rigidity is *not* sufficient.

Adding a NNN bond to the square at  $(i, j)$  constrains it to remain square in any floppy deformation: it is only allowed to rotate. This amounts to setting  $A_i = B_j$  (with the proper choice of signs). Each NNN bond provides such an equality, as illustrated in fig. 1b, in which two NNN bonds set  $B_2 = A_2 = B_4 = x$ . The rigidity of the whole lattice would correspond to having  $A_i = B_j = x$  for all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ , with  $x$  corresponding to the amount of global rotation in this case. This suggests a graph representation of realizations of the rigidity percolation procedure. The nodes represent the variables  $A_i$  and  $B_j$ , and edges between two nodes represent equalities of the corresponding variables that arise from the NNN bonds (fig. 1c). Hence, we obtain a description of rigidity percolation in terms of a random graph with two types of nodes (green  $A$  nodes and red  $B$  nodes), with edges between an  $A$ -node and a  $B$ -node present with probability  $p$ . This mapping has been employed before in the context of studying rigidity in square and cubic structures with

added NNN bonds, albeit without the probabilistic aspect of rigidity percolation [16].

Within this description, a rigid lattice is represented by a connected graph, i.e., a graph in which there is a path along edges between any given pair of nodes, corresponding to the case of all  $A_i = B_j$  with global rotation left as the only degree of freedom. Note that we use crossbars to represent the effect of the NNN bonds, because the second NNN bond on a plaquette is redundant, when only the question “rigid or not” is considered. Near the percolation transition in large systems where  $p_R \rightarrow 0$ , it is equivalent to use crossbars with probability  $p$  or to use separate NNN bonds with probability  $p/2$ , so the resulting scalings are the same.

We now derive the scaling of the probability  $\mathcal{F}(m, n, p)$  that the system is rigid, as  $m, n \rightarrow \infty$ . This can be obtained as a corollary from the results of Palásti [17], but we choose to present our own, more accessible derivation.

For the usual Erdős-Rényi model [18–20] for generating random graphs in which there are  $n$  equivalent nodes and each of the  $n(n - 1)/2$  edges connecting any two nodes is present with probability  $p$ , the probability  $\mathcal{F}_1(n, p)$  that the graph is connected can be calculated recursively from an expression due to Gilbert [18]

$$1 - \mathcal{F}_1(n, p) = \sum_{k=1}^{n-1} \binom{n-1}{k-1} \mathcal{F}_1(k, p) q^{k(n-k)}, \quad (1)$$

where  $q = 1 - p$ . Here, the probability that the graph is not connected is written as the sum over all possible sizes  $k$  of the cluster that an arbitrarily chosen node (labelled as node 1) is part of. The binomial coefficient represents the number of ways to choose the other nodes in the cluster, and the power of  $q$  denotes the probability that none of those  $k$  nodes are connected to any of the other  $n - k$  nodes. Gilbert showed that for large  $n$ ,  $\mathcal{F}_1(n, p) \rightarrow 1 - nq^{n-1}$  [18], so the connectivity threshold for the usual Erdős-Rényi model is  $p_C(L) \sim \ln n/n$ .

We adapt eq. (1) for our two-colored graph in which edges between differently colored nodes are present with probability  $p$ . Again, we consider the sum over all the possibilities for the size of the cluster having  $k$  green nodes and  $l$  red nodes that an arbitrary green node (to which we assign the variable  $A_1$ ) is part of:

$$1 = \sum_{k=1}^m \sum_{l=0}^n \binom{m-1}{k-1} \binom{n}{l} \mathcal{F}(k, l, p) q^{k(n-l)} q^{l(m-k)}. \quad (2)$$

A recurrence relation for  $\mathcal{F}(m, n, p)$  can be obtained by moving the term for  $k = m, l = n$  from the sum to the left hand side of the equation. In the Appendix we show that as  $m = n \rightarrow \infty$  the probability approaches

$$\mathcal{F}(n, n, p) \approx 1 - 2nq^n \quad \text{as } n \rightarrow \infty. \quad (3)$$

For any finite  $p$  the probability approaches unity, and the value of  $p$  needed to make half the realizations rigid ap-

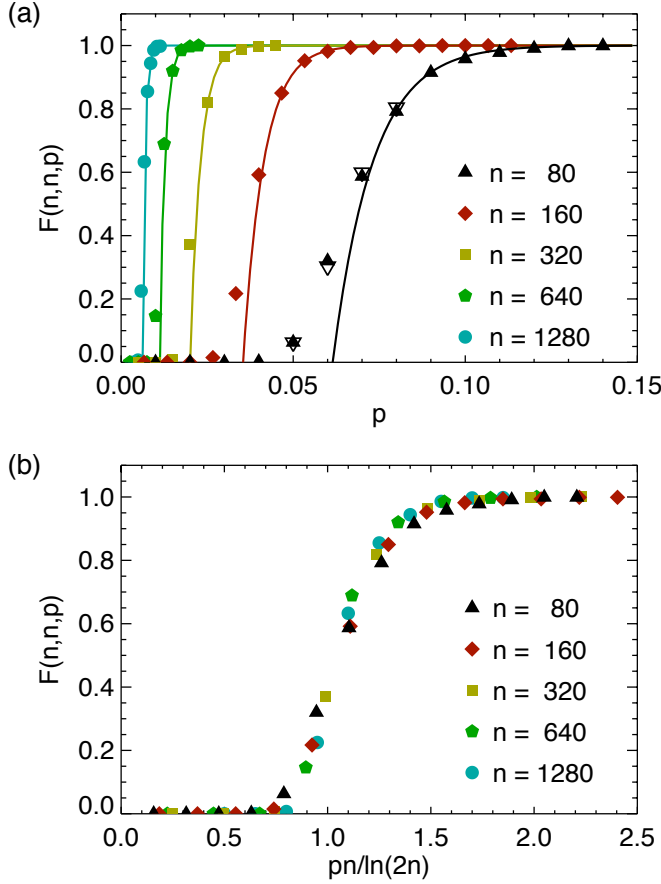


Fig. 2: (a) Numerically obtained fraction of rigid configurations  $\mathcal{F}(n, n, p)$  as a function of bond-presence probability  $p$  for various system sizes. The solid lines denote the asymptotic form  $\mathcal{F} = 1 - 2n(1 - p)^n$  [eq. (3)]. For completeness we included a few points (open triangles) obtained by numerically evaluating the recurrence relation (2) for  $n = 80$ . (b) The same data with  $p$  rescaled by  $\ln(2n)/n$ , according to eq. (4).

proaches zero as (see Appendix)

$$p_R(n) \approx \frac{\ln n}{n} + \mathcal{O}(1/n) \quad \text{as } n \rightarrow \infty. \quad (4)$$

**Numerics.** — For various values of  $n$  and  $p$ , we generated 1000 trials of the rigidity percolation procedure and analyzed the corresponding graphs using an adapted version of the Hoshen-Kopelman cluster labelling algorithm [21]. We plot the fraction  $\mathcal{F}$  of the graphs that are connected, corresponding to the fraction of rigid networks, in fig. 2a. The solid lines in this figure indicate the asymptotic form derived in eq. (3). Figure 2b shows the same data, with the  $p$ -axis rescaled by  $\ln(2n)/n$  [the 2 comes from the minimum prefactor of the  $1/n$  term in eq. (4) — see Appendix], clearly showing the scaling of  $p_R(n)$ .

**Discussion.** — The graph picture for rigidity percolation on the square lattice provides a transparent way to keep track of the effect of added constraints. The intuitive argument used in earlier works [4, 5, 11], that having

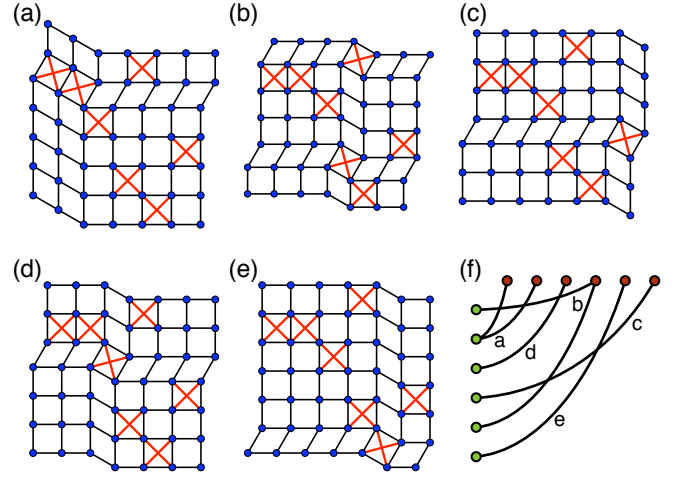


Fig. 3: Having a crossbar on each row and each column does not guarantee rigidity. (a-e) The five floppy modes of an example system. (f) The graph representation of this example system. The five connected clusters correspond to the five floppy modes in panels a-e of this figure.

at least one NNN bond in each row and column should provide rigidity, corresponds to not having any isolated nodes with no edges. Clearly this is a weaker condition than demanding the graph to be connected. This is illustrated in fig. 3, in which panels (a-e) show the five floppy modes of a lattice in which the condition of having at least one NNN bond in each row and column is satisfied. The corresponding graph is shown in fig. 3f. While counting the number of floppy modes by just looking at a picture of a lattice is far from trivial, the graph representation gives a much simpler view in which it is easy to see that there are 5 distinct connected components, so that the lattice has 5 floppy modes.

Thus, using the weaker “one-per-row” condition leads to a structural underestimation of the rigidity threshold  $p_R$ . However, this condition does provide the correct scaling with system size,  $p_R \sim \ln L/L$  [11]. This implies that the fraction of realizations that satisfy the “one-per-row” condition but are not fully connected vanishes as  $L \rightarrow \infty$ , which is consistent with the known result that just below the transition, the graph contains just one giant component and some isolated nodes [22].

As an aside, we note that the appearance of this giant component is in itself an important transition in the theory of random graphs [20]. However, for the system studied in this paper its relevance is limited. Having a giant component is not enough for rigidity to percolate, because any isolated node that is not connected to the giant component corresponds to a row or column in the lattice in which there are no crossbars, so that the two parts of the system on either side of that row or column can freely shear past each other. Thus the relevant transition is indeed that to connected graphs, as described in this letter.

The formulation of our approach in terms of the null space of the dynamical matrix implies that the floppy modes we talk about in principle only need to be floppy up to linear order, i.e. for infinitesimal deformations. However, the row and column displacements in our discussion can be straightforwardly extended to finite displacements, and thus the mapping to the graph representation is not limited to linear elasticity. This can be easily seen in fig. 3, in which the modes have a finite amplitude but are clearly zero energy modes.

In our discussion the square lattice has been assumed to be a perfect periodic lattice. However, “generic lattices”, in which only the connectivity (topology) is prescribed but bond lengths are not all equal, are often used in the discussion of rigidity theory [3, 23]. The percolation problem is still well-defined: whether or not a given structure (a set of points and rigid bars connecting them) is rigid does not depend on the precise positions of the points, but only on the connectivity and on the fact that the points do not have any positional order<sup>2</sup> [23]. This, again, seems to suggest some kind of graph theoretical approach could be helpful. However, the mapping to the graph connectivity problem that we used in this paper relies specifically on the positional order of the sites and hence does not work for the case of generic lattices. The reason is that some of the clusters of the resulting graph represent internally inconsistent constraints in this language. These clusters do not correspond to floppy modes, so that the lattice could become rigid before the graph is fully connected. The threshold probability  $p_R(n)$  for periodic lattices therefore serves as an upper bound for the  $p_R(n)$  for generic lattices, but there is no obvious lower bound, because the “one-per-row” condition that provided the lower bound on  $p_R$  in the perfect square lattice (see appendix) is no longer a necessary condition for rigidity. Clearly, a more sophisticated approach is needed to describe generic rigidity percolation. The opportunities that arise from mapping this problem to so-called *hypergraphs* are currently being explored [24].

In conclusion, we have derived an exact recurrence equation for the probability  $\mathcal{F}(L, L, p)$  that a square lattice with given size  $L$  and NNN bonds occupation probability  $p$  is rigid, and obtained the asymptotic solution to this equation in the limit of  $L \rightarrow \infty$ . Our results unambiguously prove that the rigidity percolation in the square lattice occurs at  $p = 0$  and is a *mixed first-order-second-order transition*: In the thermodynamic limit, on the one hand all lattice sites immediately become part of a percolating rigid cluster as soon as  $p$  is nonzero; on the other hand, the shear modulus increases continuously from zero at the transition [5]. Our result indicates that the system shows no fractal spatial dimension in the percolating cluster. Therefore a mean-field approach, such as the coherent potential approximation used in Ref. [12], may be sufficient for the square lattice near its rigidity threshold.

<sup>2</sup>Genericness of a set of site positions is usually defined as there not being any nontrivial algebraic equation (with rational coefficients) relating the coordinates of the sites.

Our results also verify the existence of a diverging length scale  $l^* \sim p^{-1}$ , ignoring the slowly varying  $\ln$  factor. For any given NNN bonds occupation probability  $p$ , lattices of sizes bigger than  $l^*$  are very likely to be rigid and lattices smaller than  $l^*$  are very likely to be floppy. This observation in square lattices is consistent with the cutting argument on the isostatic length scale by Wyart [25].

Counting independent constraints is key in various systems that show a rigidity transition, from jammed sphere packings [15] to rigidity percolation [1–3]. Mapping the *rigidity problem into the connectivity problem of random graphs* allows to draw inspiration from the vast body of work on (random) graphs to gain insight in the rigidity problem. This approach has led us to an exact expression for the probability that a square lattice is made rigid by randomly adding next nearest neighbor bonds. We speculate that this idea can be applied to a wider range of models for random media, and advance our understanding of disordered materials.

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**Appendix: The infinite system size limit.** — To obtain the thermodynamic limit  $m, n \rightarrow \infty$ , we follow Gilbert’s strategy of deriving a lower and upper bound to  $\mathcal{F}(n, n, p)$  and showing that these are in close agreement [18]. The upper bound on  $\mathcal{F}(n, n, p)$  is given by the lower bound on  $1 - \mathcal{F}(n, n, p)$  that is set by the probability that at least one of the  $2n$  nodes is not connected to any other node. Denoting by  $E_i$  the event that node  $i$  is not connected to any other node, which has probability  $q^n$ , we use a Bonferroni inequality [26] to obtain

$$1 - \mathcal{F}(n, n, p) \geq P\left(\bigcup_i E_i\right) \geq \sum_i P(E_i) - \sum_{i < j} P(E_i \cap E_j). \quad (5)$$

The first term on the right hand side equals  $2nq^n$ . The second term is of order  $(nq^n)^2$  and can therefore be ignored as  $n \rightarrow \infty$ , so that the upper bound becomes

$$\mathcal{F}(n, n, p) \leq 1 - 2nq^n \quad \text{as } n \rightarrow \infty. \quad (6)$$

The lower bound is obtained directly from the recurrence relation (2). We have, using  $\mathcal{F}(k, l, p) \leq 1$  and writ-

ing  $x = q^{m-2k}$ ,

$$\begin{aligned}
1 - \mathcal{F}(m, n, p) &= \sum_{k=1, l=0}^{m, n} ' \binom{m-1}{k-1} \binom{n}{l} \mathcal{F}(k, l, p) q^{nk} x^l \\
&\leq \sum_{k=1, l=0}^{m, n} \binom{m-1}{k-1} \binom{n}{l} q^{nk} x^l - 1 \\
&= \sum_{k=1}^m \binom{m-1}{k-1} q^{nk} (1+x)^n - 1 \\
&= \sum_{k=1}^m \binom{m-1}{k-1} (q^k + q^{m-k})^n - 1,
\end{aligned} \tag{7}$$

where in the first line the prime indicates the term  $(k, l) = (m, n)$  is to be excluded from the sum. In the second line the sum is extended to include the term  $(k, l) = (m, n)$  which is then corrected for by subtracting 1, and in the third line we used the binomial expansion. Now we can read off the leading terms: The terms for  $k = 1, m-1, m$  together give  $\mathcal{F}(n, n, p) \geq 1 - 2nq^n - \mathcal{O}(n^2 q^{2n-2})$ , while all other terms only give contributions of order  $(nq^n)^2$  or higher. Hence we find convergence of this lower bound with the upper bound in eq. (6), and conclude that

$$\mathcal{F}(n, n, p) \rightarrow 1 - 2nq^n \quad \text{as } n \rightarrow \infty. \tag{8}$$

This asymptotic form holds for finite  $p$  and tells us how  $\mathcal{F}$  approaches unity as bigger systems are considered. Note that the scaling of  $p_R$ , defined as the value of  $p$  where  $\mathcal{F}(n, n, p) = 1/2$ , does not immediately follow from this because  $p_R$  vanishes as  $n \rightarrow \infty$ . However, we can obtain bounds on  $p_R$  by considering where the bounds on  $\mathcal{F}$  cross  $1/2$ , as long as we evaluate them keeping all orders of  $nq^n$ . Equating the right hand side of eq. (5) to  $1/2$  and solving for very large  $n$  we obtain  $q^n = 1/(2n)$ , which gives a lower bound on  $p_R$  of

$$p_R \geq \frac{\ln 2n}{n} \quad \text{as } n \rightarrow \infty. \tag{9}$$

The upper bound on  $p_R$  follows from equating the right hand side of eq. (7) to  $1/2$ . Numerically, we find again a solution of the form  $q^n = 1/(\beta n)$ , but this time with  $\beta \approx 4.93$ , so that

$$\frac{\ln 2n}{n} \leq p_R \lesssim \frac{\ln 4.93n}{n}. \tag{10}$$

From the result of the more elaborate proof by Palásti [17] one can derive that

$$p_R = \frac{\ln(2n/\ln 2)}{n} \approx \frac{\ln 2.89n}{n},$$

which falls nicely within the bounds we derived by simpler means. In any case the leading order behavior is given by

$$p_R = \frac{\ln n}{n} + \mathcal{O}(1/n) \quad \text{as } n \rightarrow \infty. \tag{11}$$

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